Renormalisation group approach to multifractal structure in random resistor networks just above the percolation threshold

1989 J. Phys. A: Math. Gen. 22 1111

(http://iopscience.iop.org/0305-4470/22/8/024)

J. Phys. A: Math. Gen. 22 (1989) 1111-1120. Printed in the UK

Renormalisation group approach to multifractal structure in random resistor networks just above the percolation threshold

Takashi Nagatani†, Mitsuharu Ohki‡ and Motoo Hori‡

† College of Engineering, Shizuoka University, Hamamatsu 432, Japan
‡ Department of Applied Physics, Tokyo Institute of Technology, Tokyo 152, Japan

Received 10 May 1988, in final form 22 September 1988

Abstract. A renormalisation group method is presented to analyse the multifractal structure of the current distribution in random linear resistor networks just above the percolation threshold. The recursion relation for the current distribution is given under the renormalisation transformation. The distribution function of the current is derived with the use of the recursion relation. The fragmentation of the current is described as a random multiplicative process. An infinite set of exponents is calculated to scale each of the moments of the current distribution. The α-f spectrum is derived from the Legendre transform of these exponents. Our exponents obtained by the relatively small-cell renormalisation are very close to the previous simulation data.

1. Introduction

Recently, there has been increasing interest in the critical behaviour of random resistor networks. Different properties of such systems are found to be described by different critical exponents or fractal dimensions. It has been found very recently that electrical properties of self-similar resistor networks should be characterised by an infinite set of exponents (Rammal *et al* 1985a, b, de Arcangelis *et al* 1985a, b, 1986). Nagatani (1987a), Fourcade and Tremblay (1987) and Meir and Aharony (1988) have studied the multifractal structure of the current distribution in random self-similar resistor networks. In many cases, specific members of families of fractal dimensions represent geometrical and physical substructures of the underlying self-similar structure. For example, the kth moment of the currents in the percolating network is directly related to the number of singly connected bonds ($k \to \infty$), the resistance ($k = 2$) and the number of the backbone or current-carrying bonds ($k = 0$). The fact that an infinite set of exponents is necessary to characterise completely the properties of self-similar resistor networks has analogues in most fields, such as turbulence (Mandelbrot 1974, Benzi *et al* 1984), diffusion-limited aggregation (Halsey *et al* 1986b, Amitrano *et al* 1986, Nagatani 1987b), localisation (Ioffe *et al* 1985, Kohmoto *et al* 1987) and dynamical system (Hentschel and Procaccia 1983, Halsey *et al* 1986a).

De Arcangelis *et al* (1985a, b, 1986) introduced a simple hierarchical model in order to discuss the voltage distribution in self-similar resistor networks analytically. Their model is constructed from a deterministic fractal lattice. Although deterministic fractal models have been very useful (Mandelbrot and Given 1984, Nagatani 1986a, b, 1987c), they lack one of the basic features of the natural statistical fractals. They are not random, and therefore the correlation length exponent ν cannot be derived from the hierarchical model. Nagatani (1986c) has presented the regular-random fractal model which was closely connected with the position-space renormalisation. The model can present the correlation length exponent and satisfy Coniglio's relation (1982). Nagatani (1987a) derived the infinite set of exponents of the current distribution by making use of a position-space renormalisation method. Meir and Aharony (1988) found the multifractal exponents on the dilute Wheatstone bridge by using the cumulant real-space renormalisation group method. They investigated a difference between results on the specific non-random fractal (the Mandelbrot-Given curve) and averages over distributions of the specific random fractal (percolating cluster on the dilute Wheatstone bridge). De Archangelis *et al* have found the distribution function of current on the deterministic model. However, the distribution function of current is not analytically found in the percolating cluster on the real lattice.

In this paper, we present the renormalisation group method to derive the multifractality of current distribution. The present approach is the systematic development of the previous letter (Nagatani 1987a). We pay attention to the current distribution under the renormalisation transformation. We study how the current distribution is transformed by the renormalisation. We find the distribution function of the current by using the real-space renormalisation method. The multifractal exponents of current are derived from the current distribution function. We restrict ourselves to the bond percolation problem on the square lattice. In the percolation problem the renormalisation group method is known to present the exact percolation probability $p_c = \frac{1}{2}$ (Stanley *et al* 1982). By the use of the renormalisation method presenting the exact percolation threshold, we shall derive the current distribution and its scaling structure. In § 2 we present a random fractal model in the hierarchical lattice to derive the current distribution. One can picture to oneself the geometric structure in real-space renormalisation by making use of the hierarchical-random model. In § 3, we derive the recursion relation of the current distribution under the real-space renormalisation. We obtain the current distribution by repeated application of this recursion relation. In § 4, we find the scaling structure of the current distribution. We obtain the α-f spectrum for the current distribution. Section 5 presents the summary.

2. Hierarchical-random model

We present the hierarchical-random model to mimic the geometric structure under the real-space renormalisation. The hierarchical-random model is just constructed by the fine-graining procedure. One can make the model by the iteration method. At each iteration one replaces each bond by one of the spanning clusters appearing in the cell under the renormalisation procedures. (For details see Nagatani (1986c).) The model is known to satisfy all the conventional scaling relations between the critical exponents for cluster numbers and cluster structure (Nagatani 1986c). In the hierarchical model the real-space renormalisation procedure can be exactly applied. The model also presents a fractal geometric picture for the real-space renormalisation group. The simplest model was presented in the earlier letter (Nagatani 1986c). By this model we did not obtain the exact critical probability. Critical exponents were also poor. Here we extend the simple model to the more accurate one which presents the exact critical probability and good critical exponents.

We consider spanning and non-spanning configurations appearing in the renormalisation of the cell. We consider here the division of the lattice into cells with the scale factors $b = 2$ and 3 (see figure 1(a) and (b)) in the renormalisation procedure. The

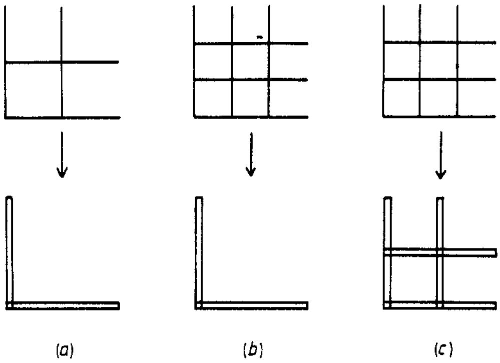

Figure 1. Illustration of the dividing and rescaling of b $(=2, 3, \frac{3}{2})$ cells on the square lattice. (a) A $b=2$ cell, (b) a $b=3$ cell, (c) the cell-to-cell transformation from a $b=3$ cell to a $b=2$ cell.

renormalisation method with the dividing cells presents the exact critical probability. We distinguish the spanning and non-spanning configurations of the cell. We make use of the spanning clusters as the generators for constructing the infinite cluster just above the percolation threshold. We consider a stepwise generation of the hierarchical-random model by using the iteration method. As an initiator, a bond is occupied with probability p_0 and unoccupied with $1-p_0$. If the bond is present, the bond is replaced with one of the spanning clusters in figure 2, and otherwise with one of the non-spanning clusters, where the occupation probability p_1 is given by $p_1 = R^{-1}(p_0)$. The function $R(p)$ indicates the renormalised occupation probability under the renormalisation transformation. The probability of which one of the spanning clusters is chosen is given by that appearing for each spanning cluster. Furthermore, the second-order generation is obtained by replacing each occupied bond with one of the spanning clusters and each unoccupied bond with one of the non-spanning clusters, where $p_2 = R^{-1}(p_1)$. The process is continued to the Nth-order generation. In the limit where N is sufficiently large, the resultant lattice approaches the percolation threshold. If $p_0 > p_c$ (where p_c is the critical probability), then the resultant lattice represents the

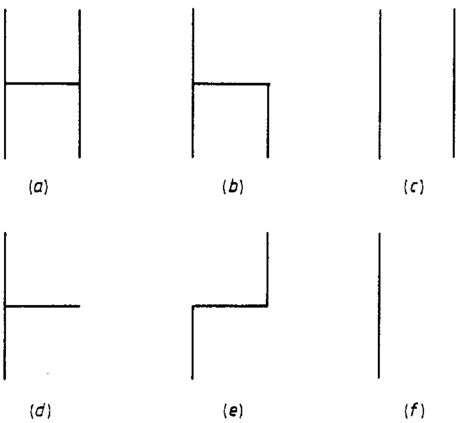

Figure 2. Spanning configurations that arise in the real-space renormalisation group for bond percolation on the square lattice.

geometric texture above the threshold. The geometric texture becomes the fractal in the range of length $L: L \ll \xi(\xi = b^N)$. The system obtained consists of the islands separated from the percolating network and a superlattice made by nodes separated by a distance $\xi = b^N$, connected by quasilinear links. The system is considered as a random fractal on a hierarchical lattice. In the lattice, the renormalisation group method can be exactly applied. For the $b = 2$ case, the geometric structure of our model is equivalent to that on the dilute Wheatstone bridge.

3. The current distribution

We pay attention to the infinite cluster on the hierarchical lattice constructed in § 2. The infinite cluster is the random fractal lattice with a hierarchical structure. We consider the distribution of the current on the hierarchical-random fractal. We give the analytical method to derive the distribution function of the current. We derive the recursion relation of the current distributions before and after the renormalisation. The current flowing on the random resistor network shows a fragmentation phenomenon (Pietronero and Siebesma 1986). Consider all the spanning configurations of the cell. Figure 2 indicates the configurations for the cell with the scale factor $b = 2$. The probability C_α that a particular spanning configuration α appears is given by

$$\begin{aligned} &C_a = p^5/R_2(p) && C_b = 4p^4(1-p)/R_2(p) && C_c = p^4(1-p)/R_2(p) \\ &C_d = 6p^3(1-p)^2/R_2(p) && C_e = 2p^3(1-p)^2/R_2(p) && C_f = 2p^2(1-p)^3/R_2(p) \end{aligned} \tag{1}$$

where $R_2(p) = 2p^5 - 5p^4 + 2p^3 + 2p^2$. $R_2(p)$ represents the probability that a cell of size $b = 2$ is connected between the entrances and the exits.

We calculate the currents carrying on each bond within the cell with the spanning configurations. When the total current carrying vertically through the cell is i, there are four bonds with the current fraction $i/2$ and a bond with no current in configuration (a) (figure 2). In configuration (b), there is one bond with the current i, two bonds with $i/3$ and one bond with $2i/3$. In this way we can obtain all the current fraction within the cell with the spanning cluster. The current $2i$ carrying through the cell (a) breaks out on each bond by the fragmentation process and results in the current i. The current $3i$ flowing through the cell (b) also breaks out on each bond and results in the current i on the two bonds in configuration (b).

We thus obtain the recursion relation between the current distributions $f_n(i)$ and $f_{n-1}(i)$ before and after the renormalisation:

$$\begin{aligned} f_n(i) &= \int \mathrm{d}j A_2(s, p)\delta(i - sj) f_{n-1}(j) \\ &= A_2(1, p) f_{n-1}(i) + A_2(\tfrac{1}{2}, p) f_{n-1}(2i) + A_2(\tfrac{1}{3}, p) f_{n-1}(3i) + A_2(\tfrac{2}{3}, p) f_{n-1}(3i/2) \end{aligned} \tag{2}$$

where $A_2(s, p)$ indicates the mean number of bonds with the current fraction s. They are given by

$$\begin{aligned} A_2(1, p) &= 1 \times C_b + 2 \times C_d + 3 \times C_e + 2 \times C_f \\ &= (10p^5 - 20p^4 + 6p^3 + 4p^2)/R_2(p) \\ A_2(\tfrac{1}{2}, p) &= 4 \times C_a + 4 \times C_c = 4p^4/R_2(p) \\ A_2(\tfrac{1}{3}, p) &= 2 \times C_b = (-8p^5 + 8p^4)/R_2(p) \\ A_2(\tfrac{2}{3}, p) &= C_b = (-4p^5 + 4p^4)/R_2(p). \end{aligned}$$

We now derive the distribution of current fraction. If a unit current is carrying through an Nth-order hierarchical lattice, the number of bonds carrying the current $i = (\frac{1}{2})^{N_1}(\frac{1}{3})^{N_2}(\frac{2}{3})^{N_3}$ is given by

$$\binom{N}{N_1}\binom{N-N_1}{N_2}\binom{N-N_1-N_2}{N_3} \times A_2(1, p_N)^{N-N_1-N_2-N_3} A_2(\tfrac{1}{2}, P_N)^{N_1} A_2(\tfrac{1}{3}, p_N)^{N_2} A_2(\tfrac{2}{3}, p_N)^{N_3} \tag{3}$$

where $p_N = R_2^{-1}(p_{N-1})$.

In the limit where N is sufficiently large, p_N approaches the critical value $p_c = \frac{1}{2}$. At the percolation threshold the current distribution is given by

$$f_N(i) = \sum_{N_1=1}^{N} \sum_{N_2=1}^{N-N_1} \sum_{N_3=1}^{N-N_1-N_2} \sum_{N_4=1}^{N-N_1-N_2-N_3} \delta(i - (\tfrac{1}{2})^{N_2}(\tfrac{1}{3})^{N_3}(\tfrac{2}{3})^{N_4}) \times (N!/N_1!N_2!N_3!N_4!) A_2(1)^{N_1} A_2(\tfrac{1}{2})^{N_2} A_2(\tfrac{1}{3})^{N_3} A_2(\tfrac{2}{3})^{N_4} \tag{4}$$

where

$$N_1 + N_2 + N_3 + N_4 = N$$

$$A_2(1) = \tfrac{26}{16} \qquad A_2(\tfrac{1}{2}) = \tfrac{8}{16} \qquad A_2(\tfrac{1}{3}) = \tfrac{8}{16} \qquad A_2(\tfrac{2}{3}) = \tfrac{4}{16}.$$

The fractal dimension of the hierarchical-random fractal is also given by

$$d_f = \log[(5p^5 - 16p^4 + 12p^3 + 4p^2)/R_2(p)]_{p=1/2}/\log 2 = \log(\tfrac{53}{16})/\log 2 = 1.7279\ldots. \tag{5}$$

Similarly we can obtain the current distribution for the scale factor $b = 3$. The current distributions $f_n(i)$ and $f_{n-1}(i)$ before and after the renormalisation are given by

$$f_n(i) = \sum_j A_3(j, p) f_{n-1}(i/j) \tag{6}$$

where $A_3(j, p)$ represents the mean number of bonds with the current fraction j. We express $A_3(j, p)$ in terms of

$$A_3(j, p) \equiv [B_0(j)/R_3(p)]p^{13} + [B_1(j)/R_3(p)]p^{12}(1-p) + [B_2(j)/R_3(p)]p^{11}(1-p)^2 + \ldots + [B_{10}(j)/R_3(p)]p^3(1-p)^{10} \tag{7}$$

where

$$R_3(p) = p^{13} + 13p^{12}(1-p) + 78p^{11}(1-p)^2 + 283p^{10}(1-p)^3 + 677p^9(1-p)^4 + 1078p^8(1-p)^5 + 1089p^7(1-p)^6 + 627p^6(1-p)^7 + 209p^5(1-p)^8 + 38p^4(1-p)^9 + 3p^3(1-p)^{10}.$$

$B_0(j)$ represents the number of bonds with the current j within the spanning cell in which all the bonds are occupied. $B_n(j)$ indicates the number of bonds with the current j within the spanning cells in which the n bonds are unoccupied and $(13-n)$ bonds occupied. One can calculate $B_n(j)$ by finding the current fraction in each configuration appearing in the renormalised cell.

We can obtain the current distribution at the percolation threshold. The fractal dimension is also given by

$$d_f = \log(\tfrac{31\,165}{4096})/\log 3 = 1.8471\ldots. \tag{8}$$

The fractal dimension obtained from the cell size $b=3$ is close to the accurate value 1.89 (Stauffer 1985).

One can easily extend (6) to the case with any scale factor b. It is given by

$$f_n(i) = \sum_j A_b(j,p) f_{n-1}(i/j). \tag{9}$$

4. Scaling structure

We shall derive the multifractal exponents of the current by using the distribution function of the current obtained in § 3. Each bond of the percolating network can be characterised by the fraction of the total current flowing through it, i.e. $\tilde{I} = I/I_{\text{tot}}$. The moments of the current distribution and corresponding exponents $\tilde{\zeta}_k$ can be defined by

$$\sum_{\tilde{I}} \tilde{I}^k f(\tilde{I}) = \left\langle \sum_{j\in\Gamma_b} \tilde{I}_j^k \right\rangle \sim L^{\tilde{\zeta}_k} \tag{10}$$

where $f(\tilde{I})$ is the number of bonds with a current fraction $\tilde{I}$, Γ_b is the set of the backbone bonds, $\langle\ \rangle$ represents the ensemble average and L indicates the system size.

One multiplies (9) by the kth-order i^k of current and sums over all the possible values of current i. The recursion relation of the moments of the current is given by

$$\left(\sum_{\tilde{I}} \tilde{I}^k f_n(\tilde{I})\right) = \left(\sum_j A(j,\tfrac{1}{2}) j^k\right)\left(\sum_{\tilde{I}'} \tilde{I}'^k f_{n-1}(\tilde{I}')\right) \tag{11}$$

where $[\Sigma_j A(j,\frac{1}{2})j^k]$ represents the kth-order moment $\langle j^k \rangle$ of the current fraction j within the cell. The relationship (11) represents a random multiplicative process of the random variable j which is the current fraction within the cell. It is the most important feature of our approach, characterising the scaling structure of the current distribution. From (11) we can construct an infinite set of exponents $\tilde{\zeta}_k$:

$$\tilde{\zeta}_k = \log\left(\sum_{\tilde{I}} \tilde{I}^k f_N(\tilde{I})\right)(\log L)^{-1} = \log\left(\sum_j A(j,\tfrac{1}{2}) j^k\right)(\log b)^{-1}. \tag{12}$$

For the $b=2$ cases, we obtain

$$\tilde{\zeta}_k = \log[(\tfrac{26}{16}) + (\tfrac{8}{16})(\tfrac{1}{2})^k + (\tfrac{8}{16})(\tfrac{1}{3})^k + (\tfrac{4}{16})(\tfrac{2}{3})^k]/\log 2. \tag{13}$$

The exponents of backbone bonds, resistance and cutting bonds are given by $\tilde{\zeta}_0 = 1.5235\ldots$, $\tilde{\zeta}_2 = 0.938\,57\ldots$ and $\tilde{\zeta}_\infty = 0.700\,43\ldots$. Also the correlation length exponent ν is given by $1/\nu = \log[\mathrm{d}R_2(p)/\mathrm{d}p]_{p=1/2}/\log 2$. This agrees with the exponent $\tilde{\zeta}_\infty$. It is found that Coniglio's relation holds. The multifractal exponents (13) were derived by Meir and Aharony (1988) on the hierarchical Wheastone bridge lattice. The coincidence is due to the equivalence between the $b=2$ cell and the unit of the Wheatstone bridge. Our method is general and can be extended to larger sizes of the cell. Similarly we can obtain the infinite set of exponents for the $b=3$ case:

$$\tilde{\zeta}_k = \log\left(\sum_j A_3(j,\tfrac{1}{2}) j^k\right)(\log 3)^{-1} \tag{14}$$

where $A_3(j, \frac{1}{2})$ is given by (7). The exponents of backbone bonds, resistance and cutting bonds are given as $\tilde{\zeta}_0 = 1.5996\ldots$, $\tilde{\zeta}_2 = 0.970\,15\ldots$ and $\tilde{\zeta}_\infty = 0.724\,80\ldots$ The reciprocal of correlation length exponent ν also agrees with the exponent of the cutting bonds:

$$\nu^{-1} = \log[\mathrm{d}R_3(p)/\mathrm{d}p]_{p=1/2}/\log 3 = \log(\tfrac{9082}{4096})/\log 3 = \tilde{\zeta}_\infty. \qquad (15)$$

In table 1, we give the numerical values of $\tilde{\zeta}_k$. Columns RG(2→1), RG(3→1), RG(3→2), H model and data indicate, respectively, the values obtained from the renormalisation group methods with the cell sizes $b = 2, 3, \frac{3}{2}$, the hierarchical model by de Arcangelis *et al* (1985a, b) and the data by simulation. Figure 3 shows the plot of the exponents $\tilde{\zeta}_k$ against k. Curves A, B and C show, respectively, the results obtained from the renormalisation group methods with the cell sizes $b = 2, 3$ and $\frac{3}{2}$.

Table 1. List of exponents from the renormalisation group approach with other sources:[a] de Arcangelis *et al* (1985 a, b); [b]de Arcangelis *et al* (1986);[c] Herrmann and Stanley (1984);[d] Herrmann *et al* (1984);[e] Lobb and Frank (1984);[f] Stauffer 1985).

	$\tilde{\zeta}_k$				
k	RG(2→1)	RG(3→1)	RG(3→2)	H model[a]	Data
−5	7.139	15.096	28.697	4.533	
−2	3.118	4.409	6.614	2.491	
−1	2.169	2.454	2.942	1.938	
0	1.523	1.599	1.729	1.500	1.58[b], 1.62[c]
1	1.142	1.183	1.253	1.188	1.196[b]
2	0.938	0.970	1.024	0.991	0.976[b], 0.97[d,e]
3	0.831	0.860	0.908	0.877	0.858[b]
4	0.775	0.802	0.848	0.815	0.784[b]
5	0.744	0.770	0.815	0.783	
6	0.727	0.753	0.797	0.766	
10	0.704	0.729	0.772	0.751	
∞	0.700	0.724	0.766	($\frac{3}{4}$)	0.75[f]

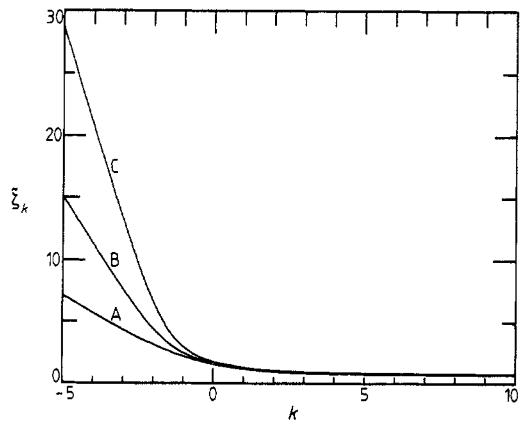

Figure 3. The plot of the exponents $\tilde{\zeta}_k$ of the moments of the current distribution against k. Curves A, B and C indicate, respectively, the results obtained from the renormalisations with the scale factors $b = 2, 3$ and $\frac{3}{2}$.

The values of exponents are effectively improved with the larger cell $b=3$. One can take into account more possible configurations with the larger cell. The possible configurations of the spanning clusters for $b=3$ are very many more than those for $b=2$. The values obtained with $b=3$ give better results in comparison with $b=2$. Accordingly, as k approaches larger negative values, the average $\langle \tilde{I}^k \rangle$ is governed by a configuration with the smaller possible current. For large negative k, the result obtained with small-size renormalisation may be poor. To improve the result for large negative k, one may take into account the averaging procedure for $k \to -\infty$ proposed by Meir and Aharony (1988).

We consider the multifractal analysis of the self-similar resitor network. In general, for each value of k, there is a corresponding distinct value of $\tilde{I}^* = \tilde{I}(k)$ which locates the peak value of the product $\tilde{I}^k f(\tilde{I})$. In the context of a finite-size scaling approach, let us now make the following scaling ansätze for $\tilde{I}(k)$ and $f(\tilde{I}(k))$:

$$\tilde{I}(k) \sim L^{\tilde{\zeta}_\infty - \alpha_k} \qquad f(\tilde{I}(k)) \sim L^{f_k}. \tag{16}$$

Table 2. List of numerical values of α and f obtained from the renormalisation group method with the scale factors $b=2$, 3 and $\frac{3}{2}$.

	α_k			f_k		
k	$b=2$	$b=3$	$b=\frac{3}{2}$	$b=2$	$b=3$	$b=\frac{3}{2}$
−5	2.187	4.587	8.689	−0.294	−4.214	−10.915
−2	1.789	3.427	6.228	0.940	−0.997	−4.310
−1	1.499	1.980	2.802	1.370	1.199	0.905
0	1.200	1.297	1.462	1.523	1.599	1.729
1	0.977	1.017	1.085	1.419	1.476	1.572
2	0.845	0.874	0.925	1.228	1.270	1.341
3	0.776	0.802	0.847	1.059	1.094	1.152
4	0.741	0.766	0.809	0.938	0.969	1.021
5	0.723	0.748	0.790	0.858	0.887	0.936
6	0.713	0.738	0.780	0.806	0.833	0.881
10	0.702	0.726	0.768	0.723	0.749	0.795
∞	0.700	0.724	0.766	0.700	0.724	0.766

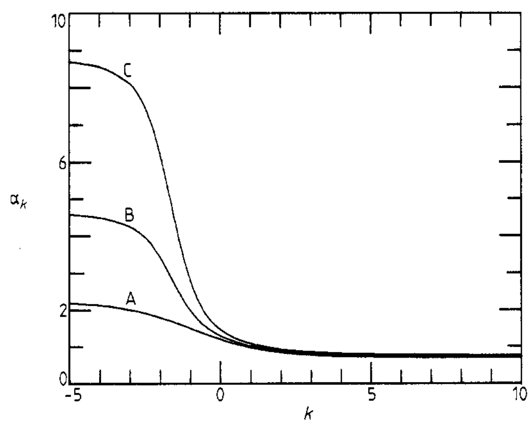

Figure 4. The plot of the function α_k against k. Curves A, B and C indicate, respectively, the results obtained from the renormalisations with the scale factors $b=2$, 3 and $\frac{3}{2}$.

The $f(\alpha)$ is the Legendre transform of $(k\tilde{\zeta}_\infty - \tilde{\zeta}_k)$:

$$(\partial/\partial k)(k\tilde{\zeta}_\infty - \tilde{\zeta}_k) = \alpha \qquad f(\alpha) = k\alpha - (k\tilde{\zeta}_\infty - \tilde{\zeta}_k). \tag{17}$$

Thus the exponent of cutting bonds is given by the maximum value of α. The highest value of the curve of the α-f spectrum gives the fractal dimension of the backbone. These relations are given by

$$\alpha_\infty = 1/\nu \qquad f_{max} = \tilde{\zeta}_0. \tag{18}$$

Table 2 indicates the numerical values of α and f. Figure 4 and 5 show the plots of α_k and f_k against k. The α-f spectra are shown in figure 6. Curves A, B and C indicate, respectively, the results obtained from the renormalisations with the cell sizes $b=2$, 3 and $\frac{3}{2}$.

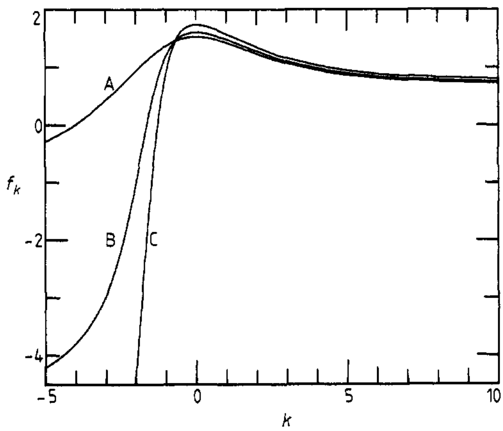

Figure 5. The plot of the function f_k against k. The curves indicate respectively the results obtained from the renormalisations with the scale factors $b=2$, 3 and $\frac{3}{2}$.

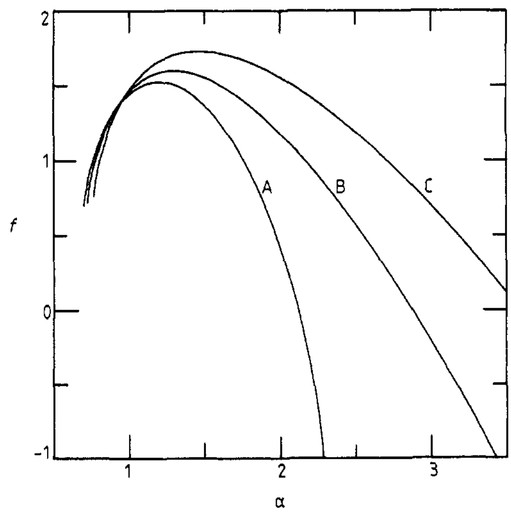

Figure 6. The α-f spectra for the multifractal structure of the current distribution. Curves A, B and C represent respectively the results of the 2×2 and 3×3 cells and the cell-to-cell transformation.

5. Summary

We present the real-space renormalisation method to analyse the scaling structure of the current distribution in the self-similar resistor network just above the percolation threshold. We obtain the general method to derive the distribution function of the current. Under the renormalisation transformation the recursion relation for the current distribution is derived. We find the current distribution with use of the recursion relation. An infinite set of exponents is calculated to describe each of the moments of the current distribution. To describe the multifractal structure the α-f spectra are derived from the Legendre transform of these exponents. Our exponents obtained by the relatively small-cell renormalisation ($b = 3$) are very close to the previous simulation data.

References

Amitrano C, Coniglio A and di Liberto F 1986 *Phys. Rev. Lett.* **57** 1016
Benzi R, Paladin G, Parisi G and Vulpiani A 1984 *J. Phys. A: Math. Gen.* **17** 3521
Coniglio A 1982 *J. Phys. A: Math. Gen.* **15** 3829
de Arcangelis L, Redner S and Coniglio A 1985a *Phys. Rev.* B **31** 4725
—— 1985b *J. Phys. A: Math. Gen.* **18** L805
—— 1986 *Phys. Rev.* B **34** 4656
Fourcade B and Tremblay A-M S 1987 *Phys. Rev.* A **36** 2352
Halsey T C, Jensen M H, Kadanoff L P, Procaccia I and Shraiman B I 1986a *Phys. Rev.* A **33** 1141
Halsey T C, Meakin P and Procaccia I 1986b *Phys. Rev. Lett.* **56** 854
Hentschel M G E and Procaccia I 1983 *Physica* D **8** 435
Herrmann H J, Derrida B and Vannimenus J 1984 *Phys. Rev.* B **30** 4080
Herrmann H J and Stanley H E 1984 *Phys. Rev. Lett.* **53** 1121
Ioffe L B, Sagdeev I R and Vinokur V M 1985 *J. Phys. C: Solid State Phys.* **18** L641
Kohmoto M, Sutherland B and Tang C 1987 *Phys. Rev.* B **35** 1024
Lobb C J and Frank D J 1984 *Phys. Rev.* B **30** 4090
Mandelbrot B B 1974 *J. Fluid Mech.* **62** 331
Mandelbrot B B and Given J A 1984 *Phys. Rev. Lett.* **52** 1853
Meir Y and Aharony A 1988 *Phys. Rev.* A **37** 596
Nagatani T 1986a *J. Phys. A: Math. Gen.* **19** L391
—— 1986b *J. Phys. A: Math. Gen.* **19** 2781
—— 1986c *J. Phys. A: Math. Gen.* **19** L1079
—— 1987a *J. Phys. A: Math. Gen.* **20** L417
—— 1987b *Phys. Rev.* A **36** 5812
—— 1987c *J. Phys. A: Math. Gen.* **20** 961
Pietronero L and Siebesma A P 1986 *Phys. Rev. Lett.* **57** 1098
Rammal R, Tannous C, Breton P and Tremblay A-M S 1985a *Phys. Rev. Lett.* **54** 1718
Rammal R, Tannous C and Tremblay A-M S 1985b *Phys. Rev.* A **31** 2662
Stanley M E, Reynolds P J, Redner S and Family F 1982 *Real-Space Renormalization* ed T W Burkhardt and J M J van Leeuwen (Berlin: Springer) p 169
Stauffer D 1985 *Introduction to Percolation Theory* (London: Taylor and Francis)